tions as they are. In either case the concept of straight line is no longer of use. That which takes its place is the concept of great circle. This is the straightest line, the line which turns neither to right nor to left, as far as motion on the surface of the sphere is concerned. The idea of similar figures has also to be abandoned. These and other differences between plane and spherical geometry will naturally be expressed by laying down a system of axioms different from those employed heretofore. This idea, that *it is the axioms which in reality makes geometry what it is*, lies at the root of the modern point of view.

We shall, then, assume a set of axioms which seem natural without at first seriously attempting to busy ourselves with the question of their completeness or independence, and upon them as a foundation we shall build up our system of spherical geometry. Later we shall examine more carefully the content of these axioms in order to put ourselves in touch with the modern treatment of the foundations of mathematics and of geometry in particular. Finally we shall touch upon the Lobatchevskyan and especially the Riemannian geometry. Incidentally there will appear novelties in arrangement of theorems, novelties in methods of proof and in method of thought. These have sprung up recently but have already taken a firm hold in science and it is to be hoped that they will find their way to some extent into the more elementary grades of instruction: for without being essentially harder they are more precise and more direct.[1]

Lecture I. The Axioms.

For the sake of brevity we shall call great circles *lines*. This will be of convenience because in many respects they resemble the straight lines in plane geometry, particularly in the Riemannian geometry. No confusion can arise, for the reason that the ordinary straight lines of plane geometry cannot exist on the spherical surface.

Axiom I. *Two points determine a line—except that to every point of the surface there corresponds one other point which taken with it will not uniquely determine a line.*

Axiom II. *Two lines invariably intersect in two points.*

In these axioms the laws of combination of the elements, lines and points, are stated. For this reason axioms I and II are called *axioms of combination*. It

1. In Europe mathematicians of note have in recent years occupied themselves with questions concerning the improvement of the presentation of Elementary Geometry. These improvements are along the following directions: The attainment of greater rigor in the treatment of problems involving limits; distinguishing more carefully what is defined, what is assumed, and what is proved; selecting modern instead of ancient methods in demonstration and especially obtaining the advantages which are to be had from introducing the important ideas of transformations, vectors, and so forth; ridding elementary work of a large number of theorems which may more properly be proved later or which are of the nature of curiosities, being of no use either practically or theoretically. See Geometrie Plane (1898) and Geometrie dans l'Espace (1899) by Niewenglowski and Gerard; also Geometrie Elementaire, Vol. 1 (1898) and Vol. 2 (1901) by Hadamard; Elementi di Geometria (1903) by F. Enriques. These authors write for those who are studying geometry for the first time and consequently do not go so far into the modern standpoint as we do in these lectures. How much of that here given might be introduced into elementary instruction without surfeiting it with difficulties is a question which the future must decide. Certainly however, a more thoughtful and less formal treatment of geometry with some discussion of such terms as "continuous" and "motion" is possible and extremely desirable. In many cases the introduction of a little more thinking and the exclusion of some useless feats of memory makes a subject easier rather than harder.

is to be noted that we have not tried to formulate any definition for "line" or for "point." They are accepted as intuitive elements and defined only so far as their properties which are stated in these and the following axioms define them.

Axiom III. *A point does not divide a line into two segments but makes of it a single piece in which the points are arranged in a natural order; that is, the words "follow," "precede," "lie between" are applicable.*

Axiom IV. *The arrangement of the points in a line is continuous.*

As these axioms tell how the points of a line are arranged relatively to one another, they are called the *axioms of arrangement.* To the first we may remark that unless some point has been chosen for a beginning and ending point, no definite arrangement of two points on a circle or any other closed curve is possible. For we may commence at any point and keep right on around the curve until we arrive at any desired point. The axiom on continuity also needs comment.

The idea of continuity is simple enough—is is merely the absence of *gaps.* By a gap in a line is generally meant that a sizable portion or segment of the line is missing. For the purposes of geometry, however, gaps which have no size, which consist in the absence of a single point, must be ruled out. To drop a perpendicular from a point to a line would be impossible if the one point where the foot of the perpendicular ought to fall were missing, and this even if every other point of the line were present. In like manner to find the middle point of a line is out of the question if the middle point is not there to find. For example, to find the middle one of a series of four marbles is a problem without solution. It is therefore necessary to formulate some definition of continuity which shall eliminate the possibility of gaps of even one single point. This may be done very simply by making use of the idea of order as stated in the third axiom.

Suppose the continuous segment AB is divided by the point C into two segments AC and CB. C is the last point of the first segment and the first point of the second. Evidently therefore, C is a point such that every point which precedes C lies in the first segment AC, and every point which follows C lies in the second segment CB. This statement is not true of any other point of the segment AB: for if X be a point of the segment AC all points between X and C follow X but do not lie in the segment CB, and similarly if X were a point of CB there would be points preceding X which however would not lie in the segment AC. Moreover if the point C were missing so as to leave a gap of a single point there would be no point of the segment AB which would agree with the statements made above concerning C. By making use of this evident property of a continuous segment a definite and scientific statement of the idea of continuity may be formulated.

Definition of continuity as understood in the above statement of axiom IV: *If the points of a segment AB are separated into two classes such that each and every point of the first class precedes each and every point of the second class, then there must exist a point C such that every point which precedes C is in the first class and every point which follows C is in the second.* This statement demands the existence of the point C and thus forestalls the possibility of a gap due to its absence. The

idea of defining the concept of continuity by this its most obvious property is due to Dedekind. We should add one thing more which is always understood when applying the definition of continuity, namely: *between any two given points of a segment there is a third point.* In other words it is impossible to take two points so close together that there shall be none between them—two "consecutive" points are, strictly speaking, impossible.*

Axiom V. *Geometrical figures may be moved freely about upon the surface of the sphere without suffering change in size or in shape.*

This is the so-called axiom of *free mobility* or *superposition.* In reality it contains a considerable number of different statements: for when analyzed, the words "size," "shape," and "moved" are no more simple than the "continuous" used in the preceding axiom. Let us therefore seek to state more precisely the ideas involved in this axiom.

Motion without change in size or shape is generally known as *rigid* motion. What, then, are the properties of rigid motion? In the first place the initial and final positions of a figure are all that need concern us. The intermediate positions through which it passes have no influence upon the final position. In other words if a figure be moved once from one position to another and then again moved so that it takes up a third position it might as well have been moved directly from the first into the third position. Any two successive motions are equivalent to a third motion. This is known as the *group* property.

Definition of a group: *When a set of transformations or changes of position is of such a kind that two successive changes of that kind result in a total change of the same kind, the set of transformations is said to form a group of transformation.*

Evidently rigid motions form just such a set, and part of Axiom V may be stated by saying—

Axiom V_1. *The set of all possible rigid motions forms a group.*

Secondly it is an obvious property of our conception of rigid motion, that

Axiom V_2. *In any motion points are carried over into points, and lines into lines.*

In the third place any point A of the spherical surface may be carried by a rigid motion into any desired point A' of the surface. Furthermore any "direction" or half-line issuing from A may be moved into any desired "direction" or half-line issuing from A'. The correspondence of the points A and A' and of the directions a and a' issuing respectively from them completely determines the motion. All other points are carried along in a perfectly definite manner when A and a go over into A' and a' respectively. The simplest motion is that in which no change of position takes place. This is called the *identical transformation:* for each point takes a position identical with its former position. We are now able to state the rest of that which is contained in Axiom V.

* It may be here noted that a segment AB has a definite last point B: but no definite next-to-the-last point. This fact which is of course unfamiliar in dealing with finite sets of quantities is of frequent occurrence in infinite sets. For instance zero may be the last number possible, but there is no next-to-the-last number. We can come as near to zero as we please but we can not get next to it for the reason that between any two given numbers there is a third number.

Axiom V_3. *Any point and direction issuing from it may be moved into any desired point and any desired direction issuing from that point—and the correspondence of these elements completely and uniquely determines the rigid motion.**

Definition. *Geometrical figures which may be carried over into one another by rigid motions are said to be congruent. The sign for congruence consists of three horizontal parallel strokes,* as $\equiv$.

$$A \text{ and } a \equiv A' \text{ and } a'.$$

POLAR COORDINATE PROOFS OF TRIGONOMETRIC FORMULAS.

By OSWALD VEBLEN, The University of Chicago.

1. Graphical, that is to say analytic geometrical methods, seem at present to be on the gain in the teaching of Trigonometry. Particularly true is this in courses conducted by the "LaboratoryMethod." This fall, I have obtained rather pleasing results by adopting a suggestion of Professor Moore to use polar coördinates. The geometric simplicity of these graphs, the sine and cosine being represented by circles and the secant and cosecant by straight lines (see Fig. 1), not only makes them attractive to the student but, unlike theCartesian graphs, makes them useful in proving theorems.

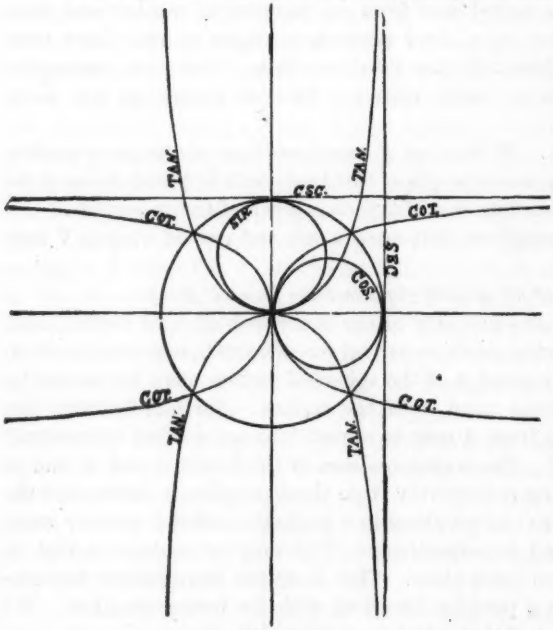


Fig. 1.

The proofs† given below, it is hoped, will demonstrate this latter point.

* The reader should convince himself that in case change of size or shape is allowed the correspondence of A and a to A' and a' will not be sufficient completely to determine the motion. The amount of distortion must also be somehow specified.

† While these proofs are probably to be found somewhere in the literature, I have not been able to find them.

That they contain some elements of simplicity I am convinced by the fact that several of my students worked out the proof of the formula for $\sin(x+y)$ with no other help than the mere suggestion to use polar coördinates. §§6, 7, 8 are based on memoranda given me by Professor Moore of work intended for his elementary calculus course. The proofs are made only for positive angles less than $\frac{1}{2}\pi$.

2. *If an angle is inscribed in a circle of unit diameter its sine is the chord of the arc subtended.*

If one side OB of the angle AOB is a diameter of the circle (see Fig. 2), then since OBA is a right angle, $\frac{AB}{OA}=\frac{AB}{1}$ is the sine of AOX. If the angle is inscribed in any other way, by a familiar theorem, it subtends the same chord as AOB.

The theorem is also true in the limiting case where one side of the angle

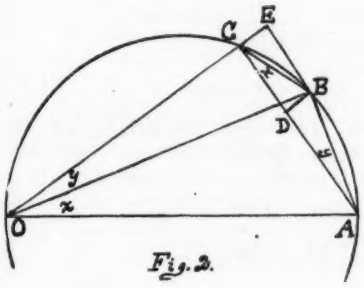


Fig. 2.

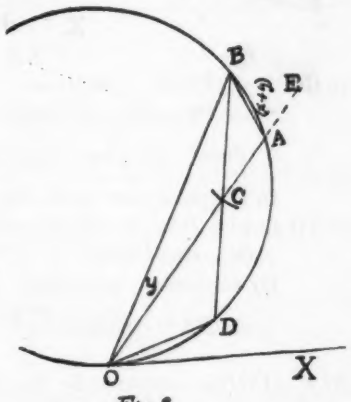


Fig. 3.

is tangent to the circle. This is the polar coördinate case and thus, in Fig. 3, $OA=\sin AOX$. We may note also that in a circle of unit diameter the length of the arc subtended by an inscribed angle is the measure of that angle in radians.

§3. Proof of the formula $\sin(x+y)=\sin x\cos y+\cos x\sin y$.

In a circle ABO (Fig. 2)* of unit diameter, let

$$\begin{aligned} \angle AOB&=x, & \therefore AB&=\sin x.\\ \angle BOC&=y, & \therefore BC&=\sin y.\\ \therefore \angle AOC&=x+y, & \therefore AC&=\sin(x+y). \end{aligned}$$

Let BD be perpendicular to AC. Then

$\angle BAC=y$ (subtending same arc as $\angle BOC$)
$\angle BCA=x$ (subtending same arc as $\angle AOB$).

*Of course, in this proof OA need not be a diameter.

$\therefore AD = AB\cos y = \sin x \cos y,$

$DC = BC\cos x = \sin y \cos x.$

Since $AC = AD + DC$, $\sin(x+y) = \sin x\cos y + \cos y \sin x$.

This proof applies directly to the polar graph (see Fig. 4) if OA is taken tangent to the circle (so that $A = O$). The present form is intended to suggest its use to those who do not care to introduce polar coördinates in a beginning course.

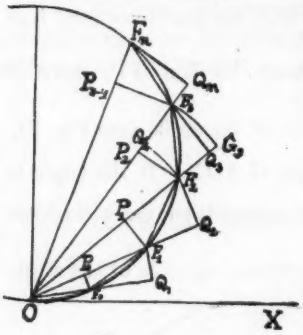


Fig. 4.

4. Proof that $\cos(x+y) = \cos x\cos y - \sin x \sin y$.

Let OAB (Fig. 2) be the polar cosine curve, *i. e.*, OA is the diameter ($=1$) of the circle OAB.

$$OB = \cos x, \quad OC = \cos(x+y), \quad BC = \sin y.$$

Let BE be perpendicular to OC. $\therefore OE/OB = \cos y$. $\therefore OE = \cos x \cos y$.

EB and CA are parallel, both being perpendicular to OC. $\therefore \angle CBE = \angle BCA = x$. $\therefore CE/CB = \sin x$. $\therefore CE = \sin x \sin y$.

Since $OC = OE - CE$, $\cos(x+y) = \cos x\cos y - \sin x \sin y$.

5. Proof that $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$.

In the polar sine circle, Fig. 3, $OA = \sin x$, $AB = \sin y$. On OB, lay off $AC = AB$ and let D be the point in which BC meets the circle.

$\therefore OC = \sin x - \sin y$.

By elementary geometry, $\angle BAE = \angle AOB + \angle ABO = x + y$.

$\therefore \angle CBA = \angle BCA = \frac{x+y}{2}$. $\therefore \angle OCD = \frac{x+y}{2}$.

Now $\angle COD$ is measured by the arc AD and hence $\angle COD = \angle ABD = \frac{x+y}{2}$

Hence OCD is an isosceles triangle and $OC = 2.OD\cos\frac{x+y}{2}$.

But $OD = \sin(\angle OBD) = \sin\left(x - \frac{x+y}{2}\right) = \sin\frac{x-y}{2}$.

$$\therefore \sin x - \sin y = OC = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}.$$

§6. First proof of the formulas

$$D_x \sin x = \cos x \quad \text{.....(1).}$$

$$\int_{x_0}^{X} \cos x\, dx = \sin X - \sin x_0 \quad \text{.....(2).}$$

The proofs in this section make use of Fig. 4, in which $\angle XOF_0 = x_0$; $\angle F_0OF_1 = \triangle x = \angle F_{i-1}OF_i$ ($1 = 1 \ldots n$); or, if one prefers to speak of the arcs,

$$\triangle x = \text{arc}F_0F_1 = \text{arc}F_1F_2 = \ldots = \text{arc}F_{n-1}F_n.$$

$F_{i-1}P_{i-1}$ is perpendicular to OF_i and F_iQ_i is perpendicular to OF_{i-1}. To prove (1), we make use of only one of the $\triangle x$ portions of the figure, for example, the third. OG_2 is taken equal to OF_2 and $OG_3=OF_3$. Then by elementary geometry

$$P_2F_3>G_2F_3=G_3F_2>F_2Q_3 \quad \text{.....(3).}$$

But if we call $\angle XOF_2=x$, $\angle F_2F_3O=x$ and $\angle F_3F_2Q_3=x+\triangle x$,

$$F_2G_3=OF_3-OF_2=\sin(x+\triangle x)-\sin x,$$
$$P_2F_3=F_2F_3\cos x=\sin\triangle x\ \cos x,$$
$$F_2Q_3=F_2F_3\cos(x+\triangle x)=\sin\triangle x\cos(x+\triangle x).$$

Hence (3) says that

$$\sin\triangle x\cos x>\sin(x+\triangle x)-\sin x>\sin\triangle x\cos(x+\triangle x) \quad \text{.....(4).}$$

$$\therefore \frac{\sin\triangle x}{\triangle x}\cos x>\frac{\sin(x+\triangle x)-\sin x}{\triangle x}>\frac{\sin\triangle x}{\triangle x}\cos(x+\triangle x).$$

Since $\underset{h=0}{L}\ \frac{\sin h}{h}=1$ and $\cos x$ is continuous, both extremes of this double inequality approach $\cos x$ as $\triangle x$ approaches zero.

Therefore the middle term approaches $\cos x$ and we have

$$D_x\sin x=\underset{\triangle x=0}{L}\ \frac{\sin(x+\triangle x)-\sin x}{\triangle x}=\cos x.$$

In this, according to our figure, $\triangle x$ was always positive. But if XOF_3 had been taken as x the same figure with similar reasoning would prove (4) for that case also.

Of course the theorem that for continuous functions, integration is the inverse of differentiation shows that (2) is a corollary of (1). But for some purposes of instruction it is worth while to compute (2) directly from the definition of an integral as the limit of a sum. Assuming the existence of a definite integral for $\cos x$ we have

$$\int_{x_0}^{X}\cos x\,dx=\underset{n=\infty}{L}\ \sum_{k=1}^{n}\cos(x_0+k\triangle x)\triangle x \quad \text{.....(5).}$$

where $\triangle x=(X-x_0)/n$, and also*

$$\int_{x_0}^{X}\cos x\,dx=\underset{n=\infty}{L}\ \sum_{k=0}^{n-1}\cos(x_0+k\triangle x)\triangle x \quad \text{.....(6).}$$

*In the familiar Cartesian figure, (5) corresponds to the inner set of rectangles and (6) to the outer.

Since $\underset{\triangle x=0}{L} \dfrac{\sin \triangle x}{\triangle x}=1$, (5) and (6) can be replaced by (7) and (8):

$$\int_{x_0}^{X} \cos x\, dx = \underset{n=\infty}{L} \sum_{k=1}^{n} \cos(x_0+k\triangle x)\sin \triangle x = \underset{n=\infty}{L}\, S_n \text{.....(7)},$$

$$\int_{x_0}^{X} \cos x\, dx = \underset{n=\infty}{L} \sum_{k=0}^{n-1} \cos(x_0+k\triangle x)\sin \triangle x = \underset{n=\infty}{L}\, S'_n \text{.....(8)},$$

and (2) will be proved if we show that $S_n<\sin X-\sin x<S'_n$.

From the quadrilateral $F_0P_0F_1Q_1$, we obtain, as we obtained (4),

$$\sin \triangle x \cos x_0 > \sin(x_0+\triangle x)-\sin x_0 > \sin \triangle x \cos(x_0+\triangle x).$$

From the second quadrilateral $F_1P_1F_2Q_2$ we similarly get

$$\sin \triangle x \cos(x_0+\triangle x) > \sin(x_0+2\triangle x)-\sin(x_0+\triangle x) > \sin \triangle x \cos(x_0+2\triangle x),$$

and so on. From the last quadrilateral we obtain, calling $X<XOF_n=x_0+n\triangle x$,

$$\sin \triangle x \cos[x_0+(n-1)\triangle x] > \sin X-\sin[x_0+(n-1)\triangle x] > \sin \triangle x \cos(x_0+n\triangle x).$$

Adding together these inequalities we see that the sum of the first terms is S'_n, and of the last terms is S_n. In the middle terms everything else cancels, leaving only $\sin X-\sin x_0$. So we have as we desired

$$S'_n > \sin X-\sin x_0 > S_n.$$

This result can be seen still more directly by noting that

$$S'_n = P_0F_1+P_1F_2+\text{.....}+P_{n-1}F_n,$$
$$S_n = F_0Q_1+F_1Q_2+\text{.....}+F_{n-1}Q_n,$$
$$OF_0=\sin x_0, \quad OF_n=\sin X.$$

If the rays centering at O be imagined to fold together like a fan from OF_0 to OF_n it is evident that S_n is less and S'_n greater than OF_n-OF_0.

§7. Second Proof of (1) and (2).

In some quarters there is a tendency to reverse the old order and present the integral calculus before the differential. The definitions of the two operations of differentiation and integration are certainly independent of each other; and whatever order may be preferred for pedagogical reasons, it is not amiss to see that in either case precisely similar methods can be used in deriving the formulas for the usual functions. That such is the case depends depends on the following theorem.*

*The proof of the first part of this theorem is made possible by the fact that any monotonic function is integrable,—a monotonic function being such that if $a<b$ either always $f(a)>$ or $=f(b)$ or always $f(a)<$ or $=f(b)$. Just as we did for the special case of $\sin x$ in the last section we can let $b-a=\triangle x$ and add up n inequalities like (9) and thus have $S_n>F(x)-Fx>S'_n$. To prove the second part divide by $b-a$ and pass to the limit as b approaches a.

If on an interval x_0, ..., X, two functions $f(x)$ and $F(x)$ have the property that for every two values of x, a and b $(x_0 \leqq a < b \leqq X)$,

$$f(a)(b-a) > F(b) - F(a) > f(b)(b-a) \quad \text{.....(9)}$$

then, first, $\int_{x_0}^{X} f(x)dx = F(X) - F(x_0)$*; second, if $f(x)$ is continuous, $D_x F(x) = f(x)$.*

In view of this theorem, to prove (1) and (2) we need only to prove the inequality

$$\cos x_0 \cdot (x - x_0) > \sin x - \sin x_0 > \cos x (x - x_0), \quad 0 \leqq x_0 < x \leqq \pi/2 \text{.....(10)}.$$

To this end we make use of the inner part of Fig. 5 in which

$$\angle XOS_0 = x_0, \quad \angle XOS = x.$$

S_0U is perpendicular to OS, S_0V to OS_0, and SO'' and S_0O'' are tangents to the sine curve OS_0B. About O'' a circle is described with radius $O''S = O''S_0$ and meeting $O''S$ in W. Since

$$\angle S_0VS = \tfrac{1}{2}\pi + (x - x_0) = \pi - [\tfrac{1}{2}\pi - (x - x_0)],$$

and $$\angle S_2O''S = \pi - 2(x - x_0) = 2[\tfrac{1}{2}\pi - (x - x_0)],$$

the circle about O'' must pass through V. Hence $SVW = \frac{1}{2}x$. From these considerations it follows that

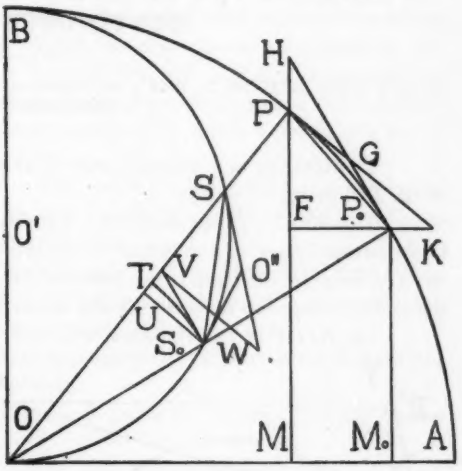


Fig. 5.

$$\sin x - \sin x_0 = TS > VS = \cos VSW \cdot WS = \cos x \cdot (S_0O'' + O''S) > \cos x \cdot \text{arc} S_0S = \cos x \cdot (x - x_0),$$

$$\text{and } TS < US = \cos USS_0 \cdot S_0S < \cos x_0 \cdot \text{arc} S_0S < \cos x_0 (x - x_0),$$

which proves (10).

§8, Second Proof of (1) and (2).

Without going into details I will add the outline of a second proof of Professor Moore's for the inequality (10) and hence for (1) and (2). This, unlike the others, is not a polar coördinate proof, but uses the unit circle. In the outer part of Fig. 5,

$$\sin x - \sin x_0 = FP = \cos FPK \cdot PK > \cos x \cdot (PG + GP_0) > \cos x \cdot \text{arc } P_0P = \cos x \cdot (x - x_0),$$

$$FP = \cos FPP_0 . P_0P = \cos\frac{x+x_0}{2} . P_0P < \cos x_0 P_0P < \cos x_0 (x - x_0).$$

$\therefore \cos x.(x-x_0) < \sin x - \sin x_0 < \cos x_0 (x-x_0)$, which is (10).

The outer part of Fig. 5 can also be used to prove that

$$\sin x - \sin x_0 = 2\cos\frac{x+x_0}{2} \sin\frac{x-x_0}{2}.$$

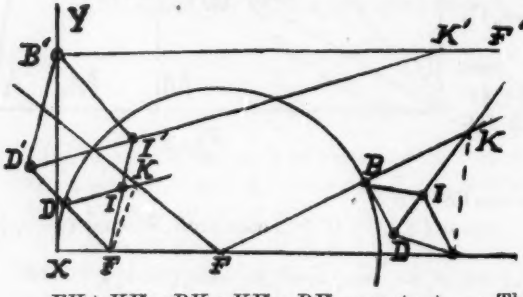
A LINKAGE FOR DESCRIBING THE CONIC SECTIONS BY CONTINUOUS MOTION.

By JOHN JAMES QUINN, Ph. B., Head of the Department of Mathematics and Manual Training, Warren High School, Warren, Pa.

The linkage is a material embodiment of the facts and conditions set forth in the following

THEOREM: *If one vertex of a movably pivoted rhombus is constrained to move in the circumference of a directing circle, while the opposite vertex is fixed in the diameter (or diameter produced), the locus of the intersection of the diagonal (produced) through the other two vertices with the radius of the directing circle is a conic.*

Let $BIDF$ be the rhombus with the vertex B moving in the circumference of the directing circle whose center is F'; F the opposite vertex fixed within (without) the diameter; and DI the diagonal produced to intersect the radius (produced) in the point K. Draw KF, and FK'. Then in the figure to the left,



$FK + KF' = BK + KF' = BF' =$ constant. $\therefore$ The locus of K is an ellipse.

In the figure to the right,

$F'K - FK = F'K - BK = BF' =$ constant. $\therefore$ The locus of K is an hyperbola.

Now suppose that the radius of the directing circle becomes infinite. Then the circumference becomes the line XY, perpendicular to the diameter (the directrix); BF becomes the line $B'F'$, parallel to the diameter, and the diagonal $D'I$ intersects it in K'.

Hence $B'K'=FK'$. $\therefore$ The locus of K' is a parabola.

COROLLARY 1. *The diagonal of the rhombus is a tangent to the curve in every position; and the second diagonal is parallel to the normal.*

COROLLARY 2. *When the diagonal is parallel to the radius it is an asymptote.*

COROLLARY 3. *If a point in one side of a movably pivoted rhombus, at a given distance from the vertex, is constrained to move in the circumference of a directing circle, and a point in the adjacent side equidistant from the same vertex is fixed in the diameter (or diameter produced), the locus of the intersection of the diagonal (produced if necessary) through the other two vertices with the radius (or radius produced) of the directing circle is a conic.*

COROLLARY 4. *A conic is the locus of a point which moves so that the ratio of its distances from a fixed circle and a fixed point in its diameter (or diameter produced) is equal to unity.*

The linkage is very easily constructed by using thin strips of wood or metal for each line in the above figure except the diameter of the directing circle, which should be strong enough to support the rest of the linkage without bending. The links representing the diagonal and the radius should be slotted, then there will be an opening at their intersection in which a pencil can be inserted to describe the curve.

The points F and F' represent the foci, one of which, the point F, should be made to slide along the diameter. Then a change in the curve due to a change in the relative position of the foci is made evident.

If the diagonal DI passes through the point F', and an extra link DE is attached to the vertex D with the point E in the diameter, but so situated that $DE=EF$, then the vertex I will describe a straight line.

For we have $FD\times FI$=constant. Consequently, if the vertex D describes a circle, the point I must describe its inverse.

The extreme simplicity of this linkage reduces the geometry of the conics to that of the rhombus and the circle.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Problem 185 was also solved by A. H. Holmes, Brunswick, Maine.

187. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Express by radicals the roots of $x^7+px^5+\frac{2}{7}p^2x^3+\frac{1}{49}p^3x+r=0$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $x=y+z$. Then the equation becomes

$$y^7+z^7+(7yz+p)[y^5+z^5+(3yz+\tfrac{5}{7}p)(y^3+z^3)+(5y^2z^2+\tfrac{5}{7}pyz+\tfrac{1}{49}p^2)(y+z)]+r=0.$$

Now x may be decomposed into two parts, y and z, in an infinite number of ways; we may, therefore, suppose y and z are such as to satisfy the condition $7yz+p=0$.

$\therefore\ yz=-\frac{1}{7}p,\ y^7+z^7=-r.$

Let $y^7=a,\ z^7=b.\quad \therefore\ a+b=-r,\ ab=-(\frac{1}{7}p)^7.$

$\therefore\ a$ and b are the roots of the equation $u^2+ru-(\frac{1}{7}p)^7=0.$

$\therefore\ a=-\frac{1}{2}r+\sqrt{[\frac{1}{4}r^2+(\frac{1}{7}p)^7]},\ b=-\frac{1}{2}r-\sqrt{[\frac{1}{4}r^2+(\frac{1}{7}p)^7]}.$

Let ω be an imaginary seventh root of unity, so that

$$\omega=\tfrac{1}{2}[A\pm\sqrt{(A^2-4)}],\ A=\tfrac{1}{7}\sqrt[7]{[\sqrt[7]{(1+3\sqrt{-3})}+\sqrt[7]{(1-3\sqrt{-3})}-\sqrt[7]{\tfrac{2}{7}}]}.$$

Then the required seven roots of the equation are

$$\sqrt[7]{a}+\sqrt[7]{b},\quad \omega\sqrt[7]{a}+\omega^6\sqrt[7]{b},\quad \omega^2\sqrt[7]{a}+\omega^5\sqrt[7]{b},\quad \omega^3\sqrt[7]{a}+\omega^4\sqrt[7]{b},$$
$$\omega^6\sqrt[7]{a}+\omega\sqrt[7]{b},\quad \omega^5\sqrt[7]{a}+\omega^2\sqrt[7]{b},\quad \omega^4\sqrt[7]{a}+\omega^3\sqrt[7]{b}.$$

188. Proposed by GUY SCHUYLER.

$$xy+ab=2ax,\quad x^2y^2+a^2b^2=2b^2y^2.$$

Solution by O. W. ANTHONY, Head of Mathematical Department, DeWitt Clinton High School, New York City.

By squaring the first equation and subtracting from the second we get $y=xa/b$ and $y=-2xa/b$. Whence easily $x=b,\ \frac{1}{2}(-1\pm\sqrt{3})b.\quad y=a(1\mp\sqrt{3})b/a.$

Also solved by G. W. Greenwood, B. A., Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.; G. W. Drake, Fayetteville, Ark.; Charles E. Barrett, Louisville, Ky.; H. F. MacNeish, A. B., Instructor in Mathematics in the University High School, Chicago, Ill.; L. E. Newcomb, Los Gatos, Cal.; G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va., and J. Scheffer, Kee Mar College, Hagerstown, Md.

GEOMETRY.

Problem 203 was also solved by Henry A. Converse, Ph. D., Instructor in Mathematics, Johns Hopkins University, Baltimore, Md.

208. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Tangents drawn to two confocal parabolae from the point on the common tangent intersect at the same angle as the axes of the parabolae.

I. Solution by G. W. GREENWOOD, A. M. (Oxon), Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

PQ is the common tangent; S, the common focus. Draw PA, PA' parallel to the axes; PT, PT' are the other tangents from P; T, T' being the points of tangency. Join PS.

Since a tangent from a point to a parabola makes the same angle with the line joining the point to the focus as the remaining tangent from the point does with the axis, we have $\angle TPA = \angle QPS = \angle T'PA'$. Hence $\angle TPT' = \angle APA'$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $y^2 = 4a(x+a)$.....(1),

$y^2\cos^2\theta + 2y\sin\theta(x\cos\theta + 2b) = 4bx\cos\theta + 4b^2 - x^2\sin^2\theta$.....(2),

be the confocal parabolae with axes inclined at an angle θ.

Let the common tangent be $y = mx + c$.....(3).

If (3) is tangent to (1), $c = a(m^2+1)/m$.

If (3) is tangent to (2), $c = b(m^2+1)/(m\cos\theta + \sin\theta)$.

$$\therefore m = \frac{a\cos\theta}{b - a\cos\theta}, \quad c = \frac{a^2 + b^2 - 2ab\cos\theta}{\sin\theta(b - a\cos\theta)}.$$

$$\therefore \text{(3) becomes } y = \frac{ax\sin\theta}{b - a\cos\theta} + \frac{a^2 + b^2 - 2ab\cos\theta}{\sin\theta(b - a\cos\theta)} \text{.....(4).}$$

Let (h, k) be the point on (4).

$\therefore y = m_1x + a(m_1^2+1)/m_1$ is tangent to (1),

$y = m_2x + b(m_2^2+1)/(m_2\cos\theta + \sin\theta)$ is tangent to (2).

But $k = m_1h + a(m_1^2+1)/m_1$.....(5),

$k = m_2h + b(m_2^2+1)/(m_2\cos\theta + \sin\theta)$....(6),

$$k = \frac{ah\sin\theta}{b - a\cos\theta} + \frac{a^2 + b^2 - 2ab\cos\theta}{\sin\theta(b - a\cos\theta)} \text{.....(7).}$$

$$\text{From (5), } m_1^2 - \frac{km_1}{a+h} + \frac{a}{a+h} = 0 \text{.....(8).}$$

$$\text{From (6), } m_2^2 + \frac{(h\sin\theta - k\cos\theta)m_2}{h\cos\theta + b} + \frac{b - k\sin\theta}{h\cos\theta + b} = 0 \text{.....(9).}$$

$\therefore m_1$ and m_2 each has two values.

From (4), $m_1 = m_2 = a\sin\theta/(b - a\cos\theta)$. The other values of m_1 and m_2 are respectively,

$$\frac{b - a\cos\theta}{(a+h)\sin\theta}, \quad \frac{(b - k\sin\theta)(b - a\cos\theta)}{a\sin\theta(h\cos\theta + b)} = \frac{b\cos\theta - h\sin^2\theta - a}{\sin\theta(h\cos\theta + b)}.$$

$$\therefore \frac{m_1 - m_2}{1 + m_1m_2} = \frac{(a^2 + b^2 - 2ab\cos\theta + 2ah\sin^2\theta + h^2\sin^2\theta)\sin\theta}{(a^2 + b^2 - 2ab\cos\theta + 2ah\sin^2\theta + h^2\sin^2\theta)\cos\theta} = \tan\theta.$$

209. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find by a geometrical method the maximum value of $\sin\theta\cos\theta\cos 2\theta$.

Solution by G. W. DRAKE, Fayetteville, Ark.

In the circle, radius 1 and center O, let $\angle AOB=\theta$, and $\angle AOC=2\theta$. From C drop a perpendicular to AO or AO produced, meeting AO in E. Then $CE=\sin 2\theta$, and $OE=\cos 2\theta$. $\sin\theta\cos\theta\cos 2\theta=\frac{1}{2}\sin 2\theta\cos 2\theta=$the area of triangle OEC. But triangle OEC is a maximum when $OE=CE$. Hence $\sin\theta\cos\theta\cos 2\theta$ is a maximum when $\sin 2\theta=\cos 2\theta$, *i. e.* when $2\theta=90-2\theta$, or when $\theta=22\frac{1}{2}°$.

Also solved by L. E. Newcomb, Los Gatos, Cal.; G. B. M. Zerr, A. M., Ph. D., Parsons, West Va. [Dr. Zerr gives the more general result $\theta=\frac{1}{8}\pi(4m+1)$.]

210. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Let ADC be a triangle with angle $C=120°$, and let the interior bisector of angle C meet AD in B. Prove that $2.CB$ is the harmonic mean between CA and CD.

Solution by M. E. GRABER, A. B., Instructor in Mathematics and Physics, Heidelberg University, Tiffin, O., and G. W. DRAKE, Fayetteville, Ark.

On AC produced through C, take a distance $CK=CD$, and join K and D. Since triangle ACB is similar to triangle AKD, $\therefore$ $BC:DK=CA:KA$, hence $2.BC=\dfrac{2.CA.DK}{KA}=\dfrac{2.CA.DK}{CA+KC}$. But because $CD=CK$, and $\angle KCD=60°$, $\therefore\angle CKD=\angle KDC=60°$, and triangle CKD is equilateral. $\therefore$ $2.BC=\dfrac{2CA.CD}{CA+CD}$. Hence $2.BC$ is the harmonic mean between CA and CD by definition.

Also solved by R. A. Wells, Bellevue College, Bellevue, Nebr.; G. W. Greenwood, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.; L. E. Newcomb, Los Gatos, Cal.; G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.; J. Scheffer, Kee Mar College, Hagerstown, Md.; E. L. Sherwood, Shady Side Academy, Pittsburg, Pa.

211. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Prove the validity of the following construction of an inscribed regular pentagon and regular decagon: Draw any two perpendicular radii of the given circle with center C. Call E the end of one radius CE and M the middle point of the perpendicular radius CM. Take the point R on CM produced through C such that $RCM=EM$. Then RC=side of inscribed regular decagon, RE=side of inscribed regular pentagon.

Solution by G. W. DRAKE, Fayetteville, Ark., and R. A. WELLS, Bellevue College, Bellevue, Neb.

Join E and M, also E and R. Let $r=CE$.

(1). $ME^2=\frac{1}{4}r^2+r^2=5r^2/4$. $ME=\frac{1}{2}r\sqrt{5}$.

$\therefore RC=RM-CM=ME-CM=\frac{1}{2}r\sqrt{5}-\frac{1}{2}r=\frac{1}{2}r(\sqrt{5}-1)=$a side of a regular decagon inscribed in a circle whose radius is r.

(2). $RE^2=RC^2+CE^2=[\frac{1}{2}r(\sqrt{5}-1)]^2+r^2=\frac{1}{4}r^2(6-2\sqrt{5})+r^2=\frac{1}{4}r(10-2\sqrt{5})$. $\therefore RE=\frac{1}{2}r\sqrt{(10-2\sqrt{5})}=$a side of a regular pentagon inscribed in a circle whose radius is r.

Also solved by G. W. Greenwood, A.B. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.; L. E. Newcomb, Los Gatos, Cal.; G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.; J. Scheffer, Kee Mar College, Hagerstown, Md., G. I. Hopkins, Manchester, N. H.

CALCULUS.

172. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Solve $x\frac{dy}{dx}=\frac{y}{y^{-1}-\log x}$.

Solution by W. W. BEMAN, A. M., Professor of Mathematics at the University of Michigan, Ann Arbor, Mich.

Writing the equation in the form $y\frac{dx}{x}+\log x\,dy=\frac{dy}{y}$, we get $y\log x=\log cy$, or, $x^y=cy$.

Also solved by G.W.Droke, Fayetteville, Ark.; M.E Graber, A.B., Instructor in Mathematics and Physics, Heidelberg University, Tiffin, O.; O.W.Anthony, DeWitt Clinton High School, New York City; G. W. Greenwood, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.; G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.

MECHANICS.

161. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

Four equal uniform smoothly jointed rods length a, and width w, form a a rhombus $ABCD$, A and C being in contact with two vertical walls b feet apart. An elastic string, natural length x, modulus λ, keeps the figure in position. The angle of friction at A and C is $\tan^{-1}p$. When the rhombus is just about to slip, find the angle A, and the angle between AB and the vertical.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Suppose the rhombus to be held in form by two strings AC, BD in a state of tension and that the rhombus is in a plane perpendicular to the walls. Let T, T' be the tensions in BD, AC; then the virtual work $T'.\delta AC+T.\delta BD=0$.

But $AC^2+BD^2=4a^2$, $\therefore$ $AC.\delta AC+BD.\delta BD=0$.

$\therefore$ $T'.BD=T.AC$ or $T'=T.AC/BD$.

Let BD make an angle θ with the vertical. Then $b=AC\cos\theta$ or $AC=b\sec\theta$, $BD=x(1+T/\lambda)=x_1$. Let R, S be the reactions at A, C. Revolving horizontally, $R+S=2T'\cos\theta$. Revolving vertically, $(R+S)p=4w$.

$$\therefore\ T'=\frac{2w}{p\cos\theta}=\frac{Tb\sec\theta}{x_1}\ \text{ or }\ T=\frac{2wx_1}{pb}.$$

$$\therefore\ x_1=x\left(1+\frac{2wx_1}{pb\lambda}\right)\ \text{ or }\ x_1=\frac{pb\lambda x}{pb\lambda-2wx}.$$

$x_1=2a\sin\frac{1}{2}A$ or $A=2\sin^{-1}(x_1/2a)$. AB makes with the vertical an angle $\frac{1}{2}B+\theta=\frac{1}{2}\pi-\frac{1}{2}A+\theta$.

162. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Show that the velocity, v, of a wave along the surface of a liquid whose

depth is not less than the wave length, λ, whose density is δ, and surface tension T, is $v^2=\frac{g\lambda}{2\pi}+\frac{2\pi T}{\lambda\delta}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ϕ denote the velocity. Taking the axis of x in the undisturbed surface and drawing the axis of y vertically downward we get, $C+p/\delta-d\phi/dt$. Since the motion is small, we can neglect the square of the velocity. The equation of continuity is $d^2\phi/dx^2+d^2\phi/dy^2=0$.

$\therefore\ \phi=\Sigma A_n e^{-my}\cos(nt-mx)$. $\therefore\ C+p/\delta=gy+\Sigma nA_n e^{-my}\sin(nt-mx)$.

At the free surface $C\delta+p$ is the excess of the pressure in the liquid over the atmospheric pressure due to the curvature and tension of the surface. Let y_0=the vertical displacement of the free surface, then d^2y_0/dx^2 is the curvature, approximately; hence at the free surface $T/\delta.d^2y_0/dx^2=C+p/\delta=gy_0+\Sigma nA_n\sin(nt-mx)$. Let $y=0$, then $dy_0/dt=d\phi/dy=-\Sigma mA_n\cos(nt-mx)$.

$\therefore\ y_0=-\Sigma(m/n)A_n\sin(nt-mx)$. $\therefore\ d^2y_0/dx^2=\Sigma(m^3/n)A_n\sin(nt-mx)$.

Substituting in the equation at the free surface we must have $m^3T/n\delta=-(m/n)g+n$ or $n^2/m^2=g/m+mT/\delta$. Now $n+2\pi v/\lambda$, $m=2\pi/\lambda$.

$\therefore\ v^2=g\lambda/2\pi+2\pi T/\lambda\delta$.

Also solved by M. E. Graber, A.B., Instructor in Mathematics and Physics, Heidelberg University, Tiffin, O.

AVERAGE AND PROBABILITY.

63. Proposed by COL. CLARKE.*

Three points are taken at random one on each of the three faces of a tetrahedron. What is the chance that a plane passing through them cuts the fourth edge? [From Williamson's *Integral Calculus*, page 410.]

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $ABCD$ be the tetrahedron, P a random point in the face ADC, Q a random point in the face ADB. Through A, P draw AF meeting DC in F; through A, Q draw AE meeting BD in E. Let $DB=a$, $DC=b$, $DE=x$, $DF=y$. On DB and DC take G and H on the same side of EF so that $EG=dx$, $FH=dy$.

The chance that a point falls in the triangle AEG is dx/a, that it falls in the triangle AFH is dy/b. In order that the plane through P, Q may cut AC and AB or the face ABC, the third point must fall on the triangle DEF. The chance of this is $(x/a)(y/b)=xy/ab$.

$\therefore$ The whole chance of the plane through P, Q cutting AB and AC is

$$p=\int_0^a\int_0^b\frac{xy}{ab}dxdy\Big/\int_0^a\int_0^b dxdy=\frac{1}{a^2b^2}\int_0^a\int_0^b xydxdy=\tfrac{1}{4}.$$

Similarly the chance of AC and BC being cut is $\frac{1}{4}$, and the chance of AB and BC being cut is $\frac{1}{4}$. $\therefore$ The chance of ABC being cut$=\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=\frac{3}{4}$.

*Vol. V, 1898, p. 120.

146. Proposed by L. C. WALKER, A.M., Professor of Mathematics, Colorado School of Mines, Golden, Col.

A random straight line crosses a given ellipse; find the chance that two points, taken at random in the ellipse, shall lie on opposite sides of the line.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let O be the center of the given ellipse, axes $OB=a$, $OC=b$. Let KL be the random chord, GH a diameter parallel to KL, and FE a diameter conjugate to GH meeting KL in Q. Let $OG=n$, $OE=m$, $OQ=z$, $\angle GOE=\phi$, $\angle BOE=\theta$. Equation to ellipse is $x^2/m^2+y^2/n^2=1$.

Area of segment $KEBL=A=2\sin\phi\int y\,dx$.

$$\therefore A=(2n\sin\phi/m)\int_z^m \sqrt{(m^2-x^2)}\,dx=\frac{n\sin\phi}{2m}[\pi m^2-2z\sqrt{(m^2-z^2)}-2m^2\sin^{-1}(z/m)]$$

Let p be the chance that both points are on the some side of KL, p' the required chance. Also, $\sin\phi=ab/mn$.

$$\therefore\ p=\int_0^{m'}\int_{-m}^{m} 2A^2\,dm\,dz/\pi^2a^2b^2\int_0^{m'}\int_{-m}^{m} dm\,dz$$

$$=\frac{1}{4\pi^2m^4}\int_0^{m'}\int_{-m}^{m}\Big[\pi^2m^4+4z^2(m^2-z^2)+4m^4\Big(\sin^{-1}\frac{z}{m}\Big)^2-4\pi m^2z\sqrt{(m^2-z^2)}$$

$$-4\pi m^4\sin^{-1}\frac{z}{m}+8m^2z\sqrt{(m^2-z^2)}\sin^{-1}\frac{z}{m}\Big]dm\,dz\div\int_0^{m'} m\,dm$$

$$=\Big(1-\frac{128}{45\pi^2}\Big)\int_0^{m'} m\,dm\div\int_0^{m'} m\,dm=1-\frac{128}{45\pi^2}.$$

$\therefore\ p'=1-p=128/45\pi^2$, the same as for a circle.

147. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

In a bag are n balls, known to be black or white, either color, *a priori*, equally likely. I draw two, which turn out to be one white and one black. I replace them and draw two more. What is the chance both are black?

Solution by W. W. LANDIS, Dickinson College, Carlisle, Pa.

This problem is open to two interpretations. We shall assume that a bag has been chosen at random from $n+1$ bags containing respectively, 0, 1, 2, ..., n black balls. The chance of any particular number of black and white balls being found is then $1/(n+1)$. The chance that the bag containing r black balls is selected and a white and black ball drawn from it then $1/(n+1).2.r/n.(n-r)/(n-1)$. Adding these probabilities from $r=1$ to $r=n-1$ we obtain as sum $\frac{1}{3}$. As this event has happened, the probabilities in the next case must be multiplied by 3. We get then for the probability of drawing two black balls from this bag

$$\frac{6}{n+1}\cdot\frac{r}{n}\cdot\frac{n-r}{n-1}\cdot\frac{r}{n}\cdot\frac{r-1}{n-1}=6[(n+1)r^3-r^4-nr^2]/(n+1)n^2(n-1)^2.$$

Adding these probabilities from $r=2$ to $r=n-1$ (as the bag containing but one black ball must be excluded) and reducing, we have, as the final result, probability required$=[3n^2-5n-2]/[10n(n-1)]$. This can easily be verified for small values of n.

PROBLEMS FOR SOLUTION.

ALGEBRA.

190. Proposed by H. H. HOLMES, Brunswick, Maine.

Find the general term of the series 2, 3, 7, 46, 2112, etc.

191. Proposed by NELSON L. RORAY, Bridgeton, N. J.

Find a number such that if it be multiplied by 2, 3, 4, 5, and 6, the cyclical order of its digits will not be changed.

GEOMETRY.

215. Proposed by M. J. NEWELL, A. M., Evanston High School, Evanston, Ill.

Construct geometrically a right triangle, given the bisectors of the acute angles.

216. Proposed by JOHN J. QUINN, Warren High School, Warren, Pa.

Find, by plane geometry, the sides of a right triangle if the hypotenuse is 35, and the side of the inscribed square is 12.

217. Proposed by G. W. DRAKE, Fayetteville, Ark.

If one of the principal axes of a cone which stands on a given base be always parallel to a given right line, the locus of the vertex is an equilateral hyperbola or a right line according as the base is a central conic or a parabola. [Exercise 40, page 94, C. Smith's *Solid Geometry*.]

218. Proposed by O. W. ANTHONY, DeWitt Clinton High School, New York City.

From a given triangle cut off an area equivalent to a given square by a line passing through a given point without the triangle.

219. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Devise a simple geometric solution of the general quadratic equation.

CALCULUS.

174. Proposed by B. F. FINKEL, 204 St. Marks Square, Philadelphia, Pa.

Integrate $\int_0^\infty \frac{\sinh px}{\sinh qx}\cos rx\,dx$, if $p^2<q^2$.

175. Proposed by M. E. GRABER, A. B., Instructor in Mathematics and Physics, Heidelberg University, Tiffin, Ohio.

Find the volume of the cono-cuneus determined by $z^2+a^2y^2/x^2=c^2$, which is contained between the planes $x=0$ and $x=a$. Ans. $\frac{1}{2}\pi c^2 a$. [Todhunter's *Integral Calculus*, p. 189, No. 28.]

MECHANICS.

165. Proposed by O. W. ANTHONY, Head of the Mathematical Department, DeWitt Clinton High School, New York City.

Find the approximate form of a tower of circular cross section 1000 feet high and having a radius of lower base 20 feet, and so constructed that all the parts of the structure shall be subject to the same stress, due to the weight of the part of the tower above.

166. Proposed by G. B.M. ZERR, A. M., Ph. D., Parsons, West Va.

If a gravitating particle of mass m be placed at the point (a, b, c) prove that the work required to move a particle of unit mass from the point (x, y, z) to an infinite distance is $m[(x-a)^2+(y-b)^2+(z-c)^2]^{-\frac{1}{2}}$. Prove also that

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\{m[(x-a)^2+(y-b)^2+(z-c)^2]^{-\frac{1}{2}}\}=0,$$

except when $x=a$, $y=b$, $z=c$.

MISCELLANEOUS.

143. Proposed by G. W. GREENWOOD, B. A. (Oxon), Professor of Mathematics and Astronomy, McKendree College, Lebanon, Ill.

If $A+B+C=180°$, show that $1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C=0$.

144. Proposed by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

Burnside, in his systematic search for all simple groups of orders from 661 to 1092 (*Proc. Lond. Math. Soc.*, 1895, pp. 333-338) overlooked the orders $792=2^3.3^2.11$ and $1008=2^4.3^2.7$. The former is immediately excluded. Discuss the latter order.

NOTES.

Beginning with January, 1904, the *Mathematical Supplement of School Science* will be published bi-monthly, as a separate magazine entitled *School Mathematics*.

At the Christmas meeting of the San Francisco Section of the American Mathematical Society the following officers were elected for the coming year: Chairman, Professor R. E. Allardice; Secretary, Professor G. A. Miller; Program Committee, Professors M. W. Haskell, Irving Stringham, and G. A. Miller.

BOOKS AND PERIODICALS.

A First Course in Infinitesimal Calculus. By Daniel A. Murray, Ph. D., Professor of Mathematics in Dahousie College, Halifax, N. S. 8vo. Cloth, xvii +439 pages. Price, $2.00. New York: Longmans, Green & Co.

We note in this work a few departures from the usual method of presentation of the Calculus. Among these departures are the following: Integration, or anti-differentiation, is presented simultaneously with differentiation; Infinite Series is relegated to the latter part of the book where also Taylor's and MacLaurin's Theorems are presented; and Indeterminate Forms are discussed in an Appendix.

In Chapter X, integration is formally taken up and considered as a process of summation. A chapter is also devoted to Differential Equations. In the Appendix is also briefly treated Hyperbolic Functions, Intrinsic Equations, and Applications to Mechanics.

In his discussion of Hyperbolic Functions, the author, following the suggestion of Professor George M. Minchin, uses *hysin*, *hycos*, *hytan*, *hycot*, *hysec*, and *hycosec*, instead of the usual *sinh*, *cosh*, etc. This notation has the advantage in that the symbols are now pronouncable, but there is a slight disadvantage resulting from the extra added letter. The notation might be abbreviated as follows: *hys*, *hyc*, *hyt*, *hyct*, *hysec*, *hycosec*. The last two not being used as much as the other four might retain the longer form.

The book contains many references for collateral reading, and a large number of problems are inserted. Many of the figures are very poorly drawn, thus marring, somewhat, the attractivensss of the book. B. F. F.

Wireless Telegraphy; Its Origins, Development, Inventions, and Apparatus. By Charles Henry Sewall, Author of "Patented Telephony," "The Future of Long-Distance Communication." 8vo. Cloth, 229 pages. Price, $2.00. New York: D. VanNostrand Co.

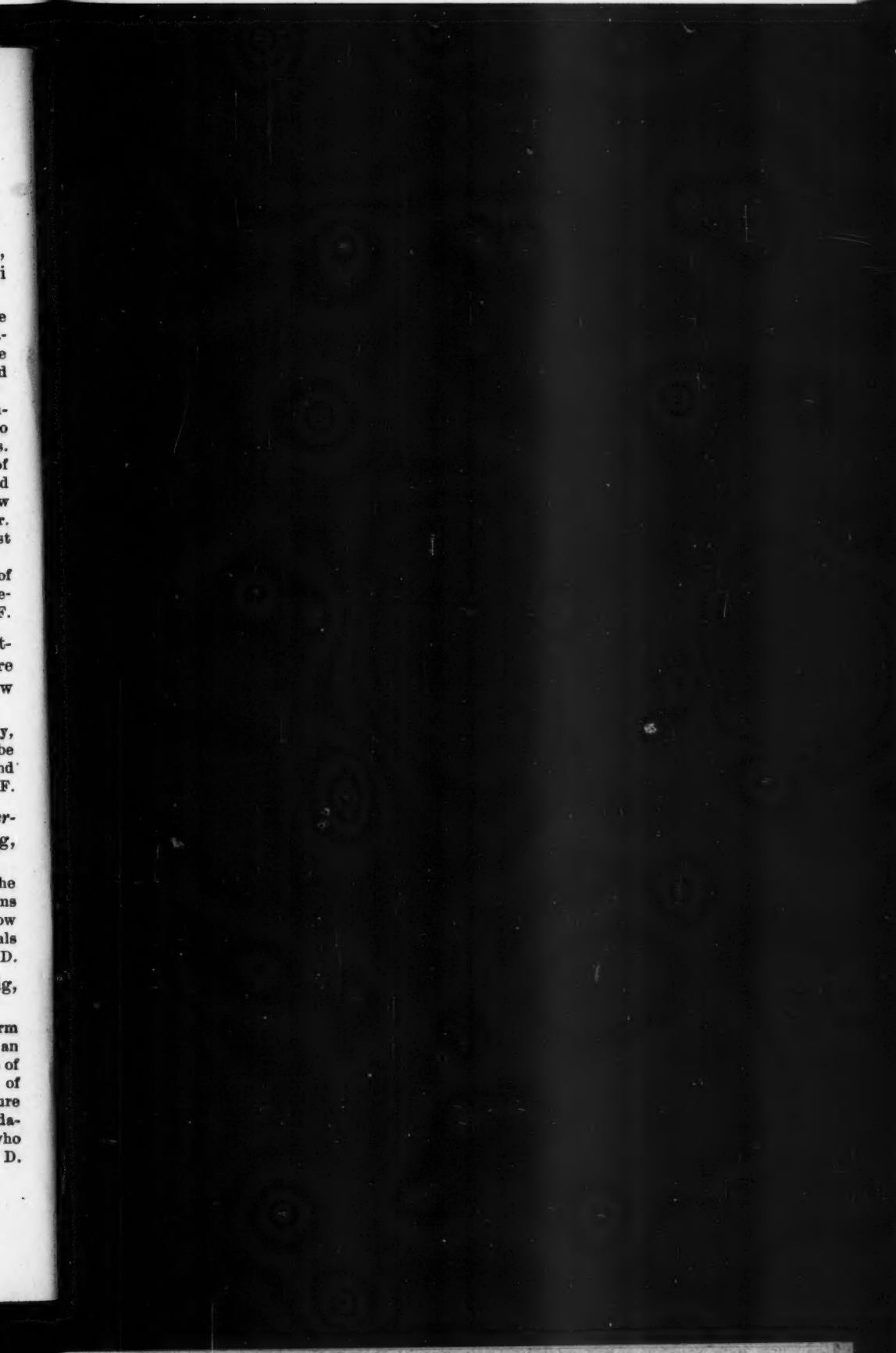
This book presents a comprehensive view of wireless telegraphy, giving its history, principles, systems in use, and the possibilities in theory and practice. The book will be found helpful to all those who are interested in this subject. The diagrams are good and the presentation is of a popular nature. B. F. F.

Einführung in die Theorie der Analytischen Funktionen einer Komplexen Veränderlichen, zweite Auflage. Von H. Burkhardt. xii+227 pages. Leipzig, Veit & Comp. 1903.

The changes that have been made in bringing out this new edition will make the text much more useful to the student, since the discouraging list of unproved theorems in Chapter III of the old edition has been replaced by ample proofs of the theorems now needed in the new proof of Cauchy's theorem, while the theorems on double integrals have been dispensed with. D.

Algebraische Analysis. Von H. Burkhardt. xii+195 pages. Leipzig, Veit & Comp. 1903.

This book, the preceding, and the 1899 volume on Elliptic Functions, together form the complete *Vorlesungen* on function-theory. The present introductory volume gives an admirable elementory account of the general theory of irrational numbers (on the basis of Dedekind's cut) and limit processes, with application to the analytic representation of the trigonometric, exponential and logarithmic functions. It is intended for mature students who have experienced in the Calculus a need of a purely arithmetical foundation, and who are ready for rigorous methods in analysis. To readers of this class, who wish a brief account of the best methods in algebra, the book will prove very welcome. D.



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SPHEICAL GEOMETRY.

By EDWIN BIDWELL WILSON, Yale University.

LECTURE II. THE ELEMENTS OF THE THEORY OF CONGRUENCE AND OF CONTINUITY.

Thera are a number of theorems on congruence and on continuity which are as self-evididentas the axioms on which they depend. The difficulty is not in giving the proof but in seeing wherein lies the necessity or possibility of proof. It must be remembered, however, that the difference between an axiom and a theorem is not that one is more readily granted as self-evident than the other. Often a theorem may be just as evidently true as is any axiom. The difference is this: An axiom is a more or less self-evident proposition arbitrarily selected selected from among a large number of such propositions to serve as a basis for future logical reasoning, whereas a theorem is a proposition which may be deduced by a number of logical steps or syllogisms from the assumed axioms. The axioms are merely a set of general statements or major premises which are necessary in order to start the chain of reasoning of which the theorems are the final results. From this it appears that a theorem possesses truth or validity only in reference to the axioms assumed and hence it is important that the axioms be as well founded as possible and in particular be not contradictory with one another. It is further clear that the rôles of the elementary theorems and of the axioms may be interchanged to a large extent, theorems becoming axioms and axioms then becoming theorems.

Theorem 1. *If in a motion one point and a direction issuing from it remain fixed all points remain fixed.*